Schematic classification of black hole geodesics using Vieta's formulae

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Preface

General Relativity is one of the most successful physical theories and one of the pillars on which stands our modern knowledge of Nature. It provides many spectacular predictions, all confirmed up to now, but the theory is also characterised by singularities, notably the cosmological one and black holes. A full quantum theory of gravity is expected to provide some solution to these, but up to now we do not have this theory at hand.

This monograph addresses the two most famous black hole solutions, Schwarzschild and Kerr ones, focusing on the motion of a test-particle on their geometries. Although a very well-known topic, we provide a view from a slightly different point of view, which is not by directly tackling and solving (mostly numerically) the geodesic equations, but trying to figure out qualitatively the shape of the trajectories by means of their turning points. To this purpose we exploit Vieta's formulas and show how much useful they are in giving an insight of the motion which takes place.

We adopt as a main reference Chandrasekar's book [Chandrasekhar, 1998] but we refer the insterested reader to other references [Carroll, 2004] [Weinberg, 1972] [Misner et al., 1973] [Frolov and Zelnikov, 2011].

Contents

Chapter 1

Introduction

When analysing the trajectories of geodesic motion in Schwarzschild or Kerr solutions, one usually faces the problem of determining the roots of a cubic or quartic function which describes the radial movement of a test-particle. These roots are special radial distances, called *turning points*, where the radial velocity vanishes and which are useful to qualitatively determine the trajectory. A formula for determining the roots of a cubic polynomial is known since the sixteenth century, Cardano's formula, but it is rather cumbersome.

The purpose of this monograph is to adopt Vieta's formulas as a tool for easily but qualitatively handle geodesics. More than that, suitable inspection of these formulas often allows to extract the main properties of the geodesic motion, without the need of a direct analytic or numerical resolution of the cubic equation. We provide the interested student with a tool which will allow him or her to clearly (hopefully) grasp the main ideas without getting lost in the mathematical details or in a cumbersome algebra.

We discuss Schwarzschild and Kerr solutions only, the most famous ones. There are of course other solutions which would deserve to be considered, but once the method outlined in this monograph will be clear, its application shall be straightforward also to other cases.

For the Schwarzschild case (due to the spherical symmetry of the solution) the motion is planar and therefore the radial motion alone is enough to completely classify the possible trajectories. On the other hand, the Kerr case is more complicated, because turning points appear also in the angular motion. This makes the variety of possible motion richer and their classification somehow trickier.

Chapter 2

Vieta's formulas

François Viète (latinized Franciscus Vieta) was a French mathematician of the XVI century. The formulas which bear his name may seem trivial at a first glance, but we hope to show their usefulness in this monograph. Let us derive them. Consider a n-degree polynomial

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \ , \tag{2.1}$$

with complex coefficient and $a_n \neq 0$. The fundamental theorem of algebra assures that the equation $P_n(x) = 0$ has n complex roots. Let $\{r_1, r_2, \cdots, r_n\}$ be these roots. Then (2.1) can be rewritten as

$$P_n(x) = a_n \left[(x - r_1)(x - r_2) \cdots (x - r_n)\right] \ . \tag{2.2}$$

Expanding the product in Eq. (2.2) and equating the result with Eq. (2.1) one obtains Vieta's formulas. Let

$$s_k := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} r_{i_1} r_{i_2} \cdots r_{i_k} \ . \tag{2.3}$$

Then, Vieta's formulas can be written as

$$s_k = (-1)^k \frac{a_{n-k}}{a_n} \ . \tag{2.4}$$

For the cubic case, which is of particular interest for our purposes, we have

$$P_3(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \ , \tag{2.5}$$

so that the three Vieta's formulas read:

$$r_1 + r_2 + r_3 = -a_2/a_3 \ , \tag{2.6}$$
$$r_1 r_2 + r_1 r_3 + r_2 r_3 = a_1/a_3 \ , \tag{2.7}$$
$$r_1 r_2 r_3 = -a_0/a_3 \ . \tag{2.8}$$

In the following chapters we employ this set of equations in order to qualitatively investigate geodesic motions in Schwarzschild and Kerr geometries. In the general non-equatorial case of geodesic motion in Kerr geometry we will see that a fourth order polynomial will drive the radial geodesic motion:

$$P_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \ . \tag{2.9}$$

In this case the four Vieta's formulas read:

$$r_1 + r_2 + r_3 + r_4 = -a_3/a_4 \ , \tag{2.10}$$

$$r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = a_2/a_4 \ , \tag{2.11}$$

$$r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 = -a_1/a_4 \ , \tag{2.12}$$

$$r_1r_2r_3r_4 = a_0/a_4 \ . \tag{2.13}$$

Chapter 3

Time-like geodesics in Schwarzschild metric

Schwarzschild solution is the first BH solution and the simplest one. Its spherical symmetry simplifies a lot the geodesic equations, basically reducing them to a one-dimensional problem (the r-motion).

3.1 Metric and geodesic equation

Schwarzschild solution (Ss) written in the traditional coordinate system has the following form:

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \ , \tag{3.1}$$

where r is the radial distance, $d\Omega^2$ is the usual S^2 line element. The parameter $M > 0$ is an integration constant, interpreted as mass.

In Ss the geodesic motion is planar, so that we can choose $\theta = \pi/2$ without loss of generality. The geodesic motion is thus described by

$$\frac{dt}{d\tau} = \varepsilon \left(1 - \frac{2M}{r}\right)^{-1} \ , \tag{3.2}$$

$$\left(\frac{dr}{d\tau}\right)^2 = \varepsilon^2 - \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right) - \delta_1 \left(1 - \frac{2M}{r}\right) \ , \tag{3.3}$$

$$\frac{d\varphi}{d\tau} = \frac{L}{r^2} \ , \tag{3.4}$$

where τ is the proper time, $\delta_1 = 1$ for massive particle geodesics and $\delta_1 = 0$ for massless particle ones. In the former cases the integration constants ε and L assume the physical meaning of specific energy and specific angular momentum, i.e. particle energy and angular momentum normalised to the particle mass. In the case $\delta_1 = 0$, ε and L represent the energy and the angular momentum of a light ray (photon).

From (3.2) we assume $r > 2M$ since we are not interested in what happens inside the horizon, where the coordinate t becomes space-like. From (3.4) one sees that the particle moves clockwisely or anti-clockwisely depending on the sign of L, but the shape of the trajectory is not affected by this, thanks to spherical symmetry of Ss. So, all the possible informations which will allow us to discriminate among different trajectories are contained in (3.3). We commence our investigation from time-like geodesics.

The roots of (3.3) represent the turning points of the motion and allow to distinguish among the possible trajectories of a test particle on the Ss background. Let

$$f(r) := r^3\dot{r}^2 = \left(\varepsilon^2 - 1\right) r^3 + 2Mr^2 - L^2 r + 2ML^2 \ . \tag{3.5}$$

An overdot represent the derivative with respect to the proper time.

3.2 The marginally bound case

We briefly discuss the marginally bound case, i.e. $\varepsilon^2 = 1$, representing a particle which is at rest at infinity. For $\varepsilon^2 = 1$ we have:

$$f(r) = 2Mr^2 - L^2 r + 2ML^2 \ , \tag{3.6}$$

so one can easily directly solve this second order equation obtaining:

$$r_\pm = \frac{L^2}{4M}\left(1 \pm \sqrt{L^2 - 16M^2}\right) \ . \tag{3.7}$$

It is the angular momentum which comes now into play:

Motion 1. If $L > 4M$, r_+ and r_- represent the perihelion and aphelion of two distinct motions. To see this, write f as

$$f(r) = 2M\left(r - r_+\right)\left(r - r_-\right) \ . \tag{3.8}$$

Since $f(r) \geq 0$, we must have $r \leq r_-$ or $r \geq r_+$. These conditions represent two separated orbits, the latter coming from infinity and having a turning point at $r = r_+$, which is thus a *perihelion*, whereas the former is an "internal" orbit with an *aphelion* at r_-. The region $r_- < r < r_+$ is forbidden. We shall call the first motion a *scattering* and the second a *trapped motion*.

Motion 2. If $L < 4M$, r_+ and r_- are complex numbers, therefore not physical. The motion in this case does not present turning points. Since $f(r)$ never vanishes for real values of r, the test-particle is doomed to cross the event horizon and fall into the singularity if $\dot{r} < 0$ or, seen in the other way around, to escape from the black hole to infinity if $\dot{r} > 0$. We shall call this kind of motion *gravitational capture*.

Motion 3. In the limiting case $L = 4M$ we have $r_+ = r_- = 4M$, i.e. a circular unstable motion since f can be written as

$$f(r) = 2M\left(r - 4M\right)^2 \ , \tag{3.9}$$

and f is always non-negative for any r. Any perturbation of this orbit causes the particle to fall into the singularity or to escape to infinity.

3.3 The unbound case $\varepsilon^2 > 1$

If $\varepsilon^2 > 1$, denote as r_1, r_2, r_3 the roots of $f(r)$. Therefore:

$$f(r) = 2M(r - r_1)(r - r_2)(r - r_3) \geq 0 \ . \tag{3.10}$$

Vieta's formulas are

$$r_1 + r_2 + r_3 = -2M \left(\varepsilon^2 - 1\right)^{-1} \ , \tag{3.11}$$

$$r_1 r_2 + r_1 r_3 + r_2 r_3 = -L^2 \left(\varepsilon^2 - 1\right)^{-1} \ , \tag{3.12}$$

$$r_1 r_2 r_3 = -2ML^2 \left(\varepsilon^2 - 1\right)^{-1} \ . \tag{3.13}$$

Since $L^2 > 0$ and $\varepsilon^2 - 1 > 0$, we have from Vieta's formulas the following possible motions:

Motion 1. Say r_1 is negative and $r_2 = a + ib$, $r_3 = a - ib$ are complex conjugate. The function f can be written as

$$f(r) = 2M \left(r + |r_1|\right) \left[(r - a)^2 + b^2\right] \ , \tag{3.14}$$

so that it is alway positive, i.e. r only increases or decreases. This is the gravitational capture.

Note that the case in which all three roots are negative is forbidden by the second Vieta's formula.

Motion 2. Say r_1 is negative and $r_- < r_+$ are positive. The function f can be written as

$$f(r) = 2M \left(r + |r_1|\right) \left(r - r_+\right) \left(r - r_-\right) \ , \tag{3.15}$$

so that, since it is always non-negative, one has $r \leq r_-$ and $r \geq r_+$. Therefore, r_- is an aphelion and r_+ is a perihelion. We have found again the scattering and the trapped motions.

Consider the limiting case $r_2 = r_3 = r_c$. At this radius an unstable circular motion takes place, since:

$$f(r) = \Delta \left(r + |r_1|\right) \left(r - r_c\right)^2 \ , \tag{3.16}$$

where we have defined $\Delta := \varepsilon^2 - 1 > 0$, for simplicity. Vieta's formulas become:

$$r_1 + 2r_c = -\frac{2M}{\Delta} \ , \qquad 2r_1 r_c + r_c^2 = -\frac{L^2}{\Delta} \ , \qquad r_1 r_c^2 = -\frac{2ML^2}{\Delta} \ . \tag{3.17}$$

Combining them it is easy to find the following relations:

$$r_1 = \frac{2Mr_c}{r_c - 4M} \ , \qquad L^2 = \frac{Mr_c^2}{r_c - 3M} \ , \tag{3.18}$$

which implies $3M < r_c < 4M$.

In the present case, the unstable orbit can be closer to the horizon than in the marginally bound case thanks to the excess energy of the particle. Employing the first of formulas (3.17), we can relate r_c to the specific energy:

$$\frac{r_c(r_c-3M)}{r_c-4M} = -\frac{M}{\Delta}\ , \qquad \Rightarrow \qquad \varepsilon^2 = \frac{(r_c-2M)^2}{r_c(r_c-3M)}\ . \tag{3.19}$$

Therefore:

$$\begin{array}{lllll} r_c \to 4M & \Rightarrow & \varepsilon^2 \to 1\ , & L^2 \to 16M^2\ , \\ r_c \to 3M & \Rightarrow & \varepsilon^2 \to \infty\ , & L^2 \to \infty\ , \end{array} \tag{3.20}$$

in the first case we reproduce the formulas of the marginally bound case.

In conclusion, given a certain specific energy $\varepsilon^2 > 1$ we determine a critical radius $3M < r_c < 4M$ from eq. (3.19) and a critical angular momentum L_c, from eq. (3.18). If $L > L_c$, the motion shall be a scattering or a trapped one, whereas for $L < L_c$ it is a gravitational capture.

3.4 The bound case $\varepsilon^2 < 1$

Finally, the last types of motions for a massive test-particle in the Ss are those corresponding to the bound case $\varepsilon^2 < 1$. In this case, the radial motion $L^2 = 0$ has a peculiarity:

$$f(r) = r^2\left[\left(\varepsilon^2-1\right)r - 2M\right]\ , \tag{3.21}$$

and the positivity of f implies

$$r \leq \frac{2M}{1-\varepsilon^2}\ , \tag{3.22}$$

i.e. the particle cannot fall from infinity. This is reasonable because at infinity the particle cannot be bound. The bound case can be obtained, for example, upon scattering processes which cause a particle to lose energy. Vieta's formulas for the bound case are

$$r_1+r_2+r_3 = 2M\left(1-\varepsilon^2\right)^{-1}\ , \tag{3.23}$$

$$r_1r_2+r_1r_3+r_2r_3 = L^2\left(1-\varepsilon^2\right)^{-1}\ , \tag{3.24}$$

$$r_1r_2r_3 = 2ML^2\left(1-\varepsilon^2\right)^{-1}\ , \tag{3.25}$$

with all the rhs positive defined. Whereas the motions of the marginally bound case and of the unbound one are practically identical (only the critical orbits change), here we expect some new orbits.

Motion 1. The three roots r_1, r_2, r_3 are real positive and suppose that $r_1 < r_2 < r_3$. Let $\Delta := 1-\varepsilon^2 > 0$. The radial function can thus be written as

$$f(r) = -\Delta\left(r-r_1\right)\left(r-r_2\right)\left(r-r_3\right)\ . \tag{3.26}$$

Since $f(r) \geq 0$, we must have that $r < r_1$ and $r_2 < r < r_3$. That is, we have a trapped motion and we have a stable motion between r_2, which is a perihelion, and r_3, which is an aphelion. This is the motion of a planet, which we shall call *bound motion* hereafter.

There are here 3 interested critical cases:

Critical case 1. If $r_1 = r_2 = r_3 = r_{ISCO}$. In this special case the radial function can be written as

$$f(r) = -\Delta\,(r - r_{ISCO})^3 \ , \tag{3.27}$$

which implies that r_{ISCO} is the radius of a circular unstable orbit, since the region $r \leq r_{ISCO}$ is allowed. Nonetheless, this radius is of the so-called innermost stable circular orbit (ISCO). Now, consider Vieta's formulas for this special case:

$$3r_{ISCO} = \frac{2M}{\Delta} \ , \qquad 3r_{ISCO}^2 = \frac{L^2}{\Delta} \ , \qquad r_{ISCO}^3 = \frac{2ML^2}{\Delta} \ . \tag{3.28}$$

These are three equations for the three unknowns r_{IS}, L and Δ. Therefore, the energy is not a parameter for this case as it was for the unbound case. The solution of the above system is very simple:

$$r_{ISCO} = 6M \ , \qquad L^2 = 12M^2 \ , \qquad \Delta = \frac{1}{9} \Rightarrow \varepsilon^2 = \frac{8}{9} \ . \tag{3.29}$$

We stress again that also the energy must be fixed in order to obtain such unstable circular orbit.

Critical case 2. If $r_1 < r_2 = r_3 = r_c$ the radial function is

$$f(r) = -\Delta\,(r - r_1)\,(r - r_c)^2 \ . \tag{3.30}$$

The region $r > r_1$ is forbidden except for $r = r_c$, which represents the radius of a stable circular orbit. The region $r < r_1$ represents a trapped motion.

Critical case 3. If $r_1 = r_2 = r_c < r_3$ the radial function becomes

$$f(r) = -\Delta\,(r - r_c)^2\,(r - r_3) \ , \tag{3.31}$$

and all the region $r < r_3$ is allowed, including $r = r_c$, which is now the radius of an unstable circular motion. Any perturbation deviates the particle from $r = r_c$ so that it either directly plunges into the singularity or performs a trapped motion.

Consider Vieta's formulas for the last two critical cases. We dub r_A the different root:

$$r_A + 2r_c = \frac{2M}{\Delta} \ , \qquad r_c\,(2r_A + r_c) = \frac{L^2}{\Delta} \ , \qquad r_A r_c^2 = \frac{2ML^2}{\Delta} \ . \tag{3.32}$$

Combining the second and the third, we find:

$$r_c = \frac{4Mr_A}{r_A - 2M} \ , \qquad \text{or} \qquad r_A = \frac{2Mr_c}{r_c - 4M} \ . \tag{3.33}$$

From the positivity of r_A and r_c we have that $r_A > 2M$ and $r_c > 4M$. From the latter we deduce that there are no circular orbits, stable or unstable, for $r < 4M$.

As we have found from Eq. (3.30), for $r_A < r_c$ the circular orbits are stable. Therefore, combining Eq. (3.33) with the condition $r_A < r_c$ it is easy to find $r_c > 6M$.

We then are in the position to conclude the following: stable circular orbits may exist for $r > 6M$. For $4M < r < 6M$ only unstable ones may exist.

Let us consider the values of the angular momentum and of the energy from eq. (3.33):

$$L^2 = \frac{Mr_c^2}{r_c - 3M} \ , \qquad \varepsilon^2 = \frac{(r_c - 2M)^2}{r_c(r_c - 3M)} \ . \tag{3.34}$$

Note that, being $r_c > 4M$, the specific energy is always $\varepsilon^2 < 1$. Indeed, when $r_c \to 4M$, we have $\varepsilon^2 \to 1$ and $L^2 \to 16M^2$, i.e. the values corresponding to the marginally bound case. The energy tends to unity also for $r_c \to \infty$. It must therefore attain a minimum in the range $(4M, \infty)$ for r_c. No surprise, this minimum corresponds to $r_c = r_{ISCO} = 6M$ and $\varepsilon^2 = 8/9$.

Therefore, if we fix the specific energy $8/9 < \varepsilon^2 < 1$, we find two possible values for r_c, one larger and one smaller than $6M$. The larger corresponds to a stable circular orbit, whereas the smaller to an unstable one if the angular momentum is the one given in eq. (3.34). If it is larger, there is no stable circular orbit, but only the trapped motion. If it is smaller, then the bound motion starts to appear

Case 2. Say r_1, r_2 are negative and r_3 is positive. A closer insepction of Vieta's formulas allows to find that this case is indeed impossible. From the first and second one gets:

$$r_3 > |r_1| + |r_2| \ , \qquad r_3 < \frac{|r_1||r_2|}{|r_1| + |r_2|} \ , \tag{3.35}$$

which is impossible. One also realises this when consider the critical case, as in the previous case.

Case 3. Say r_1, r_2 are complex conjugates and r_3 is positive. No turning points.

Chapter 4

Null-geodesics in Schwarzschild metric

In this case, $\delta_1 = 0$ and our radial equation becomes

$$\dot{r}^2 = \varepsilon^2 - \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right) , \tag{4.1}$$

so that our cubic polynomial is now defined as

$$f(r) := r^3\dot{r}^2 = \varepsilon^2 r^3 - L^2 r + 2ML^2 . \tag{4.2}$$

Its roots r_1, r_2, r_3, according to Vieta's formulas, satisfy

$$\begin{aligned} &r_1 + r_2 + r_3 = 0 , \\ &r_1 r_2 + r_1 r_3 + r_2 r_3 = -L^2/\varepsilon^2 , \\ &r_1 r_2 r_3 = -2ML^2/\varepsilon^2 . \end{aligned} \tag{4.3}$$

Necessarily we must have

1. One negative root (say r_1) and two positive ones.
2. One negative root (say r_1 again) and two complex conjugate ones, but with positive real part.

We already know that the first class corresponds to an internal motion with an aphelion plus a disconnected external motion with a perihelion. Moreover, we also know that the second class corresponds to a motion without turning points. Eliminating r_1 from the above Vieta's formulas, one obtains:

$$\begin{aligned} &(r_2 + r_3)^2 - r_2 r_3 = L^2/\varepsilon^2 , \\ &r_2 r_3 (r_2 + r_3) = 2ML^2/\varepsilon^2 . \end{aligned} \tag{4.4}$$

In the critical case $r_2 = r_3 = r_c$ one gets:

$$r_c = 3M , \tag{4.5}$$

and this circular orbit is of course unstable.

Chapter 5

Time-like geodesics in Kerr metric

In this chapter we apply Vieta's formulas to the analysis of geodesic motion in the Kerr (K) solution [Kerr, 1963]. For equatorial trajectories ($\theta = \pi/2$) the study is very simple, resembling in many aspects the one we performed in the previous chapter for Schwarzschild solution. For this reason we focus on the analysis of the more general case. It is taking into account the θ motion which renders the task more arduous. Equatorial motion in Kerr geometry is discussed in great detail in [Chandrasekhar, 1998].

5.1 Kerr metric and geodesic equations

Kerr metric in Boyer-Lindquist coordinates [Boyer and Lindquist, 1967] is written as follows:

$$ds^2 = \rho^2 \frac{\Delta}{\Sigma^2} dt^2 - \frac{\Sigma^2}{\rho^2} \left(d\varphi - \frac{2aMr}{\Sigma^2} dt \right)^2 \sin^2\theta - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 \ , \tag{5.1}$$

where M has the usual interpretation of mass of the central object, a is the specific angular momentum and the other functions are defined as follows:

$$\rho^2 := r^2 + a^2\cos^2\theta \ , \quad \Delta := r^2 - 2Mr + a^2 \ , \quad \Sigma^2 := \left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta \ . \tag{5.2}$$

The function Δ has two roots:

$$r_\pm = M \pm \sqrt{M^2 - a^2} \ , \tag{5.3}$$

the larger of them is the event horizon, whereas the smaller one is the Cauchy horizon. As in the previous chapter, we are only interested in trajectories outside the event horizon, i.e. $r > r_+$, so that $\Delta > 0$ always.

The geodesics equations are:

$$\begin{aligned}
\Delta\rho^2\dot{t} &= \Sigma^2\varepsilon - 2aMrL_z \ , && (5.4)\\
\Delta\rho^2\dot{\varphi} &= \Phi := 2aMr\varepsilon + \left(\rho^2 - 2Mr\right) L_z \csc^2\theta \ , && (5.5)\\
\rho^4\dot{r}^2 &= R := \varepsilon^2 r^4 + \left(a^2\varepsilon^2 - L_z^2 - \mathcal{Q}\right) r^2 + 2Mr\left[\mathcal{Q} + (L_z - a\varepsilon)^2\right] \\
&- a^2\mathcal{Q} - \delta_1 r^2 \Delta \ , && (5.6)\\
\rho^4\dot{\theta}^2 &= \Theta := \mathcal{Q} + \left(a^2\varepsilon^2 - L_z^2\csc^2\theta\right)\cos^2\theta - \delta_1 a^2\cos^2\theta \ , && (5.7)
\end{aligned}$$

where ε is the specific energy, L_z is the specific angular momentum in the z direction (the only component which conserves), and $\mathcal{Q}$ is Carter's constant. We indicate again the derivative with respect to the proper time as a dot. Moreover, $\delta_1 = 1$ for massive test particles or $\delta_1 = 0$ for massless ones.

Carter's constant is defined as

$$\mathcal{Q} \equiv p_\theta^2 + a^2 \cos\theta^2 (\delta_1 - \varepsilon^2) + \cot^2\theta L_z \ , \tag{5.8}$$

where p_θ is the momentum in the θ direction. Therfore, $\mathcal{Q}$ has no definite sign. It is convenient for our purposes to introduce

$$K \equiv \mathcal{Q} + (L_z - a\varepsilon)^2 \ , \tag{5.9}$$

which is always non-negative, since, using the eq. (5.8), we can write:

$$K = p_\theta^2 + \left(\frac{L_z}{\sin\theta} - a\varepsilon \sin\theta \right)^2 + \delta_1 \cos^2\theta \ , \tag{5.10}$$

which indeed is always non-negative.

Before specialising the analysis to the time-like and null geodesics, we can discuss briefly eqs. (5.4) and (5.5), which hold true both for massless as well as for massive particles.

From eq. (5.4), we assume that $\dot{t} > 0$, i.e. the particle does not go backward in time. This condition is always satisfied if $aL_z < 0$, i.e. the BH and the particle have angular momenta of opposite sign. If, on the other hand, $aL_z > 0$, then we must have:

$$\frac{a\varepsilon}{L_z} > \frac{2Ma^2 r}{\Sigma^2} \ , \qquad \text{if } a/L_z > 0 \ . \tag{5.11}$$

From eq. (5.5), we can easily determine the turning point for the φ motion, given by $\Phi = 0$. In the Ss case, there are no turning point for the latter, as you can check by setting $a = 0$ in eq. (5.5). However, the angular momentum of Kerr solution couples with that of the particle, changing this behaviour.

Note in eq. (5.5) the contribution $\rho^2 - 2Mr$, whose larger root determines the ergosphere radius:

$$r_{\text{Er}} = M + \sqrt{M^2 - a^2 \cos^2\theta} \ . \tag{5.12}$$

Casting eq. (5.5) as follows:

$$\frac{\Phi \sin^2\theta}{L_z} = r^2 - 2Mr + a^2\cos^2\theta + 2Mr \frac{a\varepsilon}{L_z} \sin^2\theta \ , \tag{5.13}$$

we deduce that, if a and L_z have the same sign, there is no turning point in the φ motion outside the ergosphere. Vice-versa, if $a/L_z < 0$ there might be a turning point at the radius given by the largest root of the above function.

For $a/L_z > 0$, i.e. when the particle has the same angular momentum as the BH's one, it appears to exist a turning point inside the ergosphere. However, this is not the case because inside the ergosphere if $\dot{\varphi} = 0$ then $\varepsilon < 0$. Therefore, $\dot{\varphi}$ is never vanishing inside the ergosphere.

A special case is the equatorial motion, for which $\theta = \pi/2$ (one must fix also $\mathcal{Q} = 0$) for which one has the turning point:

$$r_\varphi = 2M \left(1 - \frac{a}{L_z}\varepsilon\right) . \tag{5.14}$$

Therefore, when $a/L_z < 0$ we have a turning point in the φ equatorial motion which certainly takes place beyond the ergosphere, since $r_\varphi > 2M$. The BH compels the test particle to co-rotate.

The φ motion is exhausted by the above analysis, therefore in the following sections we shall concentrate only on the r and θ motion.

With the choice $\delta_1 = 1$ let us rewrite (5.6) and (5.7) as follows:

$$\begin{aligned} R &= \left(\varepsilon^2 - 1\right) r^4 + 2Mr^3 + \left[a^2\left(\varepsilon^2 - 1\right) - L_z^2 - \mathcal{Q}\right] r^2 + 2MKr - a^2\mathcal{Q} \,, \\ \Theta &= \mathcal{Q} + \left[a^2\left(\varepsilon^2 - 1\right) - L_z^2 \csc^2\theta\right] \cos^2\theta \,. \end{aligned} \tag{5.15}$$

Note the remarkable result by Carter on the separability of the geodesic equations in Kerr metric, which allows to treat separately the r and θ motions.

Now the radial equation is a fourth order polynomial in r. If $a = 0$, you can factorize an r and, provided the identification $L^2 = \mathcal{Q} + L_z^2 = K$, you obtain again eq. (3.5). Also the colatitude equation is of the fourth order, in $\cos\theta$. Let us rewrite it as:

$$\Theta \sin^2\theta = -a^2\left(\varepsilon^2 - 1\right)\cos^4\theta + \left[a^2\left(\varepsilon^2 - 1\right) - L_z^2 - \mathcal{Q}\right]\cos^2\theta + \mathcal{Q} \,. \tag{5.16}$$

Again, if $a = 0$ and identifying $L^2 = \mathcal{Q} + L_z^2$, you obtain the same geodesic colatitudinal equation as in the Schwarzschild case.

The above eq. (5.16) for θ is a quadratic equation for $\cos^2\theta$. Therefore, it can be easily solved by means of the usual formula.

Now, the marginally bound case $\varepsilon^2 = 1$ is somehow interesting, though rather speculative, since the above equations become again cubic. Let us discuss it first.

5.2 The marginally bound case $\varepsilon^2 = 1$

Let us rewrite the function R and Θ for $\varepsilon^2 = 1$:

$$R = 2Mr^3 - \left(L_z^2 + \mathcal{Q}\right) r^2 + 2MKr - a^2\mathcal{Q} \,, \tag{5.17}$$

$$\Theta = \mathcal{Q} - L_z^2 \cot^2\theta = p_\theta^2 \,, \tag{5.18}$$

where in the last equality we used the definition eq. (5.8) for the Carter constant $\mathcal{Q}$. Therefore, $\Theta \geq 0$, as it should be. We now discuss some specific cases.

Motion 1. If $\mathcal{Q} = L_z = 0$ (i.e. $p_\theta = 0$) one has that the motion happens for constant θ, no turning point for φ and since $R = 2Mr\left(r^2 + a^2\right) > 0$, this means that the motion is a free fall into the BH (or an escape), dragged by a which makes $\dot{\varphi} \neq 0$ even if $L_z = 0$.

Motion 2. If just $L_z = 0$ then the particle revolves around the BH traveling in the θ direction. At the same time, since $\Phi = \pm 2aMr$, it also turns around the symmetry axis dragged by the BH angular momentum.

For the radial motion, we have

$$R = 2Mr^3 - \mathcal{Q}r^2 + 2MKr - a^2\mathcal{Q} . \tag{5.19}$$

Here, $\mathcal{Q}$ has to be positive, since Θ is non-negative. Let us use Vieta's formulas:

$$r_1 + r_2 + r_3 = \mathcal{Q}/2M , \tag{5.20}$$

$$r_1 r_2 + r_1 r_3 + r_2 r_3 = K , \tag{5.21}$$

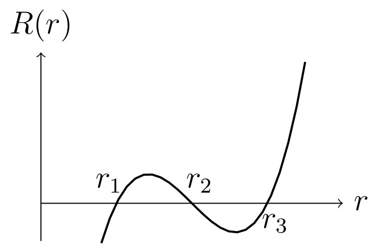
$$r_1 r_2 r_3 = a^2\mathcal{Q}/2M . \tag{5.22}$$

All the three rhs's are strictly positive (we have already considered $\mathcal{Q} = 0$ in the first paragraph). We have here two possibilities:

1. All three roots are positive and say $r_1 < r_2 < r_3$. In this case, we can write R as follows:

$$R = 2M(r - r_1)(r - r_2)(r - r_3) . \tag{5.23}$$

Since $R \geq 0$, we deduce that there is a motion confined between $r_1 < r < r_2$ and another motion which comes from infinity, attains a perihelion in r_3 and returns to infinity. All this, remember, while rotating in the θ and ϕ directions simultaneously. There are three special cases. If $r_1 = r_2 = \bar{r} < r_3$, then $\bar{r}$ is a stable circular orbit.



If $r_1 < r_2 = r_3 = \bar{r}$, then $\bar{r}$ is an unstable circular orbit. Finally, if $r_1 = r_2 = r_3 = \bar{r}$, then the latter is an unstable circular orbit.

2. The second possibility is to have one positive root and two negative ones or one positive root and two complex conjugate ones. In both cases the only possible motion is the external one with a perihelion.

Motion 3. The case $\mathcal{Q} = 0$ and $L_z \neq 0$ is not admissible because Θ must be non-negative. The only exception is the equatorial motion, for which $\cot\theta = 0$. From the non-negativity of Θ, we deduce:

$$\cot^2\theta \leq \frac{\mathcal{Q}}{L_z^2} , \qquad \Rightarrow \qquad \frac{\pi}{2} - \arctan\frac{\sqrt{\mathcal{Q}}}{|L_z|} \leq \theta \leq \frac{\pi}{2} + \arctan\frac{\sqrt{\mathcal{Q}}}{|L_z|} . \tag{5.24}$$

therefore the motion is confined, oscillating about the equatorial plane between the above two angles.

The radial motion is the same as the one presented for Motion 2, since only the coefficient of r^2 changes (it becomes $\mathcal{Q} + L_z^2$), but remains positive so does not change the classification of the roots based on Vieta's formulas.

5.3 The unbound case $\varepsilon^2 > 1$

In this case we have to deal, in general, with four Vieta's formulas, but there are two special cases, which involve $\mathcal{Q} = 0$ and for which the equations for the turning points decrease of one order.

5.4 Motion 1: $\mathcal{Q} = L_z = 0$

In this case we get

$$R/r = \left(\varepsilon^2 - 1\right) r^3 + 2Mr^2 + a^2 \left(\varepsilon^2 - 1\right) r + 2Ma^2\varepsilon^2 \ , \tag{5.25}$$

$$\Theta = a^2 \left(\varepsilon^2 - 1\right) \cos^2\theta \ . \tag{5.26}$$

The form of Θ clearly suggest that the motion is either equatorial or there are no turning points in the θ motion, so the particle revolves around the BH in the θ direction. Also, R/r is a sum of strictly positive terms and therefore never vanishes. This motion is the usual radial motion, i.e. a fall or an escape from the BH (which was expected, since there is no angular momentum).

5.5 Motion 2: $\mathcal{Q} = 0$ and $L_z \neq 0$

The angular equation is:

$$\Theta \sin^2\theta = \cos^2\theta \left[a^2 \left(\varepsilon^2 - 1\right) \sin^2\theta - L_z^2\right] \ . \tag{5.27}$$

From which it appears clear that $\cos\theta = 0$ is a possible solution, i.e. the equatorial motion is possible.

On the other hand, if $\cos^2\theta \neq 0$, then we can write

$$\Theta \tan^2\theta = a^2 \left(\varepsilon^2 - 1\right) \sin^2\theta - L_z^2 \ . \tag{5.28}$$

Since the rhs must be positive, the motion is confined around the equatorial plane:

$$\sin^2\theta \geq \frac{L_z^2}{a^2(\varepsilon^2 - 1)} \ , \tag{5.29}$$

However, in order for this to be possible, $L_z^2 < a^2 \left(\varepsilon^2 - 1\right)$, otherwise the above equation has no solutions. If this condition does not hold true, then $\mathcal{Q}$ cannot be set zero a priori.

We can again reduce the equations for the radial turning points of one order:

$$R/r = \left(\varepsilon^2 - 1\right) r^3 + 2Mr^2 + \left[a^2 \left(\varepsilon^2 - 1\right) - L_z^2\right] r + 2M \left(L_z - a\varepsilon\right)^2 \ , \tag{5.30}$$

and since $L_z^2 < a^2 \left(\varepsilon^2 - 1\right)$ we have the same radial motion of the previous case, i.e. without turning points.

5.6 General motion

Let us start from the angular motion. Since $\mathcal{Q} \neq 0$, the equatorial motion is not possible. Using $\mu = \cos^2\theta$, we can cast the angular equation (5.16) as follows:

$$\Theta(1-\mu^2) = -a^2\left(\varepsilon^2-1\right)\mu^2 + \left[a^2\left(\varepsilon^2-1\right) - L_z^2 - \mathcal{Q}\right]\mu + \mathcal{Q}\,, \tag{5.31}$$

which is an upside-down parabola, i.e. with a maximum. We are interested in where this parabola crosses the μ axis, especially if in the range $0 < \mu < 1$. First of all, we have the following Vieta's formulas:

$$\mu_1 + \mu_2 = 1 - \frac{L_z^2 + \mathcal{Q}}{a^2(\varepsilon^2-1)}\,, \qquad \mu_1\mu_2 = -\frac{\mathcal{Q}}{a^2(\varepsilon^2-1)}\,. \tag{5.32}$$

And, we have four possibilities:

1. $a^2(\varepsilon^2-1) < L_z^2 + \mathcal{Q}$ and $\mathcal{Q} < 0$. This is forbidden by the fact that Θ must be non-negative. Vieta's formulas denote an inconsistency, since the product of the roots is positive whereas their sum is negative.

2. $a^2(\varepsilon^2-1) > L_z^2 + \mathcal{Q}$ and $\mathcal{Q} < 0$. We may have two complex conjugate roots, with positive real part, which correspond to a motion with no turning points in θ.

On the other hand, two positive roots say $\mu_1 < \mu_2$ are also admissible. Then $\mu_1 < \mu < \mu_2$, in order for Θ to be positive and $\mu_1 < \cos^2\theta < \mu_2$. This represents motions confined in two sectors symmetric wrt the equatorial plane: $\arccos\sqrt{\mu_2} < \theta < \arccos\sqrt{\mu_1}$ and $\pi - \arccos\sqrt{\mu_1} < \theta < \pi - \arccos\sqrt{\mu_2}$. Note that the first Vieta's formula ensures that $\mu_2 < 1$ and $\mu_1 < 1$.

3. $a^2(\varepsilon^2-1) < L_z^2 + \mathcal{Q}$ and $\mathcal{Q} > 0$. We have one positive and one negative root, say μ_+ and μ_-. The motion is confined about the equatorial plane, since $0 < \mu < \mu_+$, which implies that $\arccos\sqrt{\mu_+} < \theta < \pi - \arccos\sqrt{\mu_+}$.

4. $a^2(\varepsilon^2-1) > L_z^2 + \mathcal{Q}$ and $\mathcal{Q} > 0$. Same as the previous case.

Vieta's formulas for the radial motion are:

$$r_1 + r_2 + r_3 + r_4 = -\frac{2M}{\varepsilon^2-1}\,, \tag{5.33}$$

$$r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = a^2 - \frac{L_z^2+\mathcal{Q}}{\varepsilon^2-1}\,, \tag{5.34}$$

$$r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 = -2M\frac{K}{\varepsilon^2-1}\,, \tag{5.35}$$

$$r_1r_2r_3r_4 = -\frac{a^2\mathcal{Q}}{\varepsilon^2-1}\,. \tag{5.36}$$

Here also we must distinguish among some cases:

1. $\mathcal{Q} > 0$. We can have three positive roots and a negative one, or one positive root and three negative ones.

2. If $a^2 (\varepsilon^2 - 1) > L_z^2 + \mathcal{Q}$ and $\mathcal{Q} < 0$, then there are no turning points for the radial motion.

5.7 The bound case $\varepsilon^2 < 1$

Chapter 6

Null geodesics in Kerr metric

With the choice $\delta_1 = 0$ let us rewrite (5.6) and (5.7) as follows:

$$R = \varepsilon^2 r^4 + \left(a^2\varepsilon^2 - L_z^2 - \mathcal{Q}\right) r^2 + 2Mr\left[\mathcal{Q} + (L_z - a\varepsilon)^2\right] - a^2\mathcal{Q} \,, \tag{6.1}$$

$$\Theta := \mathcal{Q} + \left(a^2\varepsilon^2 - L_z^2 \csc^2\theta\right)\cos^2\theta \,. \tag{6.2}$$

Note in the radial equation the absence of the cubic term. The colatitude equation can be rewritten as:

$$\Theta \sin^2\theta = -a^2\varepsilon^2 \cos^4\theta + \left(a^2\varepsilon^2 - L_z^2 - \mathcal{Q}\right)\cos^2\theta + \mathcal{Q} \,. \tag{6.3}$$

Of course one can solve the above equation for $\cos^2\theta$, using the standard formula. The solution is somehow cumbersome, as you can verify. There is perhaps more insight if we use again Vieta's formulas:

$$x_1 + x_2 = 1 - \frac{L_z^2 + \mathcal{Q}}{a^2\varepsilon^2} \,, \tag{6.4}$$

$$x_1 x_2 = -\frac{\mathcal{Q}}{a^2\varepsilon^2} \,, \tag{6.5}$$

where we have defined $x := \cos^2\theta$. Now, we can establish the following: if $\mathcal{Q} > 0$ then one root must be necessarily negative and we discard it, say x_2, since the cosine would be imaginary. We are then left with $\cos^2\theta_1 = x_1$ which gives two solutions, two turning points, in $[0, \pi]$.

On the other hand, if $\mathcal{Q} < 0$ then we may have x_1 and x_2 both negative or complex conjugate, in which case there would be no turning point, or both positive, in which case we would have four turning points.

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